POLAND, BĘDLEWO 2006

11th ICFEI

Problems and Remarks

1. Remark. *Dynamical systems generated by two maps*

Let $\delta_j\colon I \to I$, $j = 1, 2$, be continuous maps of the interval $I = [-1, 1]$ into itself satisfying:

$$1^\circ \text{ all } \delta_j \text{ do not decrease;} \quad 2^\circ\ \mathcal{R}(\delta_1) \cap \mathcal{R}(\delta_2) = \{0\}; \quad 3^\circ\ \mathcal{R}(\delta_1) \cup \mathcal{R}(\delta_2) = I$$

with $\mathcal{R}(\delta_j)$ a range of δ_j. The semigroup Φ_δ generated by δ_1, δ_2 consists of al maps $\delta_J\colon I \to I$ of the form $\delta_J = \delta_{j_n} \circ \cdots \circ \delta_{j_1}$, where $J = (j_1, \ldots, j_n)$ is an arbitrary multi-index with $j_k = 1$ or $j_k = 2$. A sequence $(t_1, \ldots, t_n, \ldots)$ of points $t_k \in I$ is called *orbit* if for all $k = 1, 2, \ldots$ we have

$$t_{k+1} = \delta_{j_k}(t_k), \qquad j_k \in \{1, 2\}. \tag{$*$}$$

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be arbitrary disjoint closed subsets in I and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. An orbit $(t_1, t_2, \ldots)$ is called $\mathcal{T}$*-guiding* if in $(*)$ $j_k = 1$ as $t_k \in \mathcal{T}_2$ and $j_k = 2$ as $t_k \in \mathcal{T}_1$. This notion plays a crucial role when studying various forms of the solvability of general linear functional equations. For example, when describing the kernel of the Cauchy type operator $CF := F(\delta_1 + \delta_2) - F(\delta_1) - F(\delta_2)$ with the above δ_1, δ_2 the result follows immediately if we note that the maximal value of any element $F \in \ker C$ spreads along $\mathcal{T}$-guiding orbits, where $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ and $\mathcal{T}_j = \{t \mid \delta_j'(t) = 0\}$, see [B. Paneah, Funct. Anal. Appl., **37** (2003), 46–60]. Finally, *attractor* $\mathcal{A}$ in Φ_δ is a collection of points $x \in I$ such that for any point $t \in I$ there is a $\mathcal{T}$-guiding orbit $(t, \delta_{j_1}(t), \ldots)$ converging to x. The main problem (solution of which finds immediately many applications) is as follows: given maps δ_1, δ_2 and sets $\mathcal{T}_1, \mathcal{T}_2$ to describe all attractors of the dynamical system Φ_δ. A particular solution of the problem is given in the above mentioned paper.

Boris Paneah

2. Remark.

During the 44th International Symposium on Functional Equations held in Louisville in May, 2006, Janusz Brzdęk asked on all self-mappings of a given semigroup satisfying the equation

$$f(x) + f(y + f(y)) = f(y) + f(x + f(y)). \tag{1}$$

Recently, Marcin Balcerowski from Katowice proved some results on (1) as well as on the more general equation

$$f(x) + f(y + g(y)) = f(y) + f(x + g(y)). \tag{2}$$

Among them the following can be proved.

Theorem

Let G be a group and let $g\colon G \to G$. Assume that the group $\langle g(G)\rangle$ generated by $g(G)$ is G. Let H be an Abelian group. Then $f\colon G \to H$ satisfies (2) if and only if it is affine, that is

$$f(x) = a(x) + b, \qquad x \in G$$

with an additive $a\colon G \to H$ and a $b \in H$.

COROLLARIES

1. Let G be an abelian group and let $f: G \to G$. Assume that $\langle f(G) \rangle = G$. Then f is a solution of (1) if and only if it is affine.

2. A function $f: \mathbb{R} \to \mathbb{R}$ is a continuous solution of (1) if and only if

$$f(x) = ax + b, \qquad x \in \mathbb{R},$$

with some $a, b \in \mathbb{R}$.

3. A function $f: \mathbb{C} \to \mathbb{C}$ is an analytic solution of (1) if and only if

$$f(z) = az + b, \qquad z \in \mathbb{C},$$

with some $a, b \in \mathbb{C}$.

Witold Jarczyk

3. Problem.

Let $D \subset \mathbb{R}^2$ be an open region. *Determine the general solution of*

$$k(x+y) = f(x)g(y) + h(y) \quad ((x,y) \in D). \tag{1}$$

More exactly, determine all $f: D_1 \to \mathbb{R}$, $g, h: D_2 \to \mathbb{R}$, $k: D_+ \to \mathbb{R}$ satisfying (1), where

$$\begin{aligned} D_1 &:= \{ \ x \mid \exists y : (x,y) \in D \ \}, \\ D_2 &:= \{ \ y \mid \exists x : (x,y) \in D \ \}, \\ D_+ &:= \{ \ x+y \mid (x,y) \in D \ \}. \end{aligned} \tag{2}$$

BACKGROUND.

I solved equation (1) (*Proc. Amer. Math. Soc.* **133** (2005), 3227-3233) when k is *locally nonconstant* (not constant on neighbourhood of any point in D_+; called *philandering* by Lundberg, Sablik et al.)

No other assumption. The *problem is to eliminate this one assumption.*

Why is the equation (1) *interesting?*

$$f(x+y) = f(x)g(y) + h(y) \quad (k = f) \tag{3}$$

is fundamental to characterising power means among quasiarithmetic means.

For $k = h$ the equation

$$k(x+y) = f(x)g(y) + h(y) \quad ((x,y) \in D) \tag{1}$$

i.e.

$$h(x+y) = f(x)g(y) + h(y) \quad ((x,y) \in D) \tag{4}$$

played an important role in comparison of utility representations (Gilányi-Ng-Aczél, *J. Math. Anal. Appl.* **304** (2005), 572–583).

Of course, also *the Pexider equations*

$$k(x+y) = f(x)g(y) \quad ((x,y) \in D) \tag{5}$$

and

$$k(x+y) = f(x) + h(y) \quad ((x,y) \in D) \tag{6}$$

are particular cases of (1).

As is known, (6) *can be solved by extension*, that is there exist $F, H, K: \mathbb{R} \to \mathbb{R}$ satisfying

$$F = f \text{ on } D_1, \quad H = h \text{ on } D_2, \quad K = k \text{ on } D_+$$

and

$$K(u+v) = F(u) + H(v) \quad \text{for} \quad (u,v) \in \mathbb{R}^2.$$

Surprisingly, *for* (5) *such an extension is in general, not possible* (possible only if k is nowhere zero on D_+), as Fulvia Skoff showed by *counterexample.*

By a constructive method, Baker, Aczél and Skoff *found the general solutions of* (5). Similarly, *if k is not locally nonconstant, extension would not work* in general for (1), *another (constructive?) method* would be needed to find the general solution of (1).

János Aczél

4. Remark and Problem. *On the stability of the Hermite–Hadamard inequality*

The convexity of a continuous real function $f: I \to \mathbb{R}$ defined on an open interval $I \subseteq \mathbb{R}$ is characterized by both sides of the well-known Hermite–Hadamard inequality, i.e., we have the following

Fact 1

The following three assertions are equivalent:

(i) *f is convex;*

(ii)

$$\frac{1}{y-x}\int_x^y f(t)dt \le \frac{f(x)+f(y)}{2} \qquad (x, y \in I,\ x<y);$$

(iii)

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)dt \qquad (x, y \in I,\ x<y).$$

For a proof and further generalizations see the book of Niculescu and Persson [4] and the paper [1].

Related to ε-convexity, we have the next (easy to verify)

Fact 2

Assume that f is ε-convex in the following sense

(i)* $\quad f(tx+(1-t)y) \le tf(x) + (1-t)f(y) + \varepsilon \quad (x, y \in I,\ t \in [0,1]).$

Then

(ii)*

$$\frac{1}{y-x}\int_x^y f(t)dt \le \frac{f(x)+f(y)}{2} + \varepsilon \qquad (x, y \in I,\ x<y);$$

(iii)*

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)dt + \varepsilon \qquad (x, y \in I,\ x<y).$$

Conversely, if (ii)* *and* (iii)* *hold then* f *is* 4ε*-convex.*

Proof. Assume that f is ε-convex. Then, integrating (i)* with respect to t over $[0,1]$, one obtains (ii)*. To deduce (iii)*, observe that (i)* implies

$$f\left(\frac{x+y}{2}\right) \le \frac{f(tx+(1-t)y)+f(ty+(1-t)x)}{2} + \varepsilon \quad (x,y\in I,\, t\in[0,1]).$$

Integrating this inequality with respect to t over $[0,1]$, one arrives at (iii)*.

If (ii)* and (iii)* hold then

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + 2\varepsilon \qquad (x,y\in I).$$

Now, using the result of Ng and Nikodem [3], the 4ε-convexity of f follows. $\square$

A problem presented at the 5th Katowice–Debrecen Winter Seminar in Będlewo was if any of the inequalities (ii)* or (iii)* implies the $c\varepsilon$-convexity of f for some positive constant c. By a recent paper of Nikodem, Riedel and Sahoo [5], the answers to both of these questions are negative, i.e., neither (ii)* nor (iii)* imply the $c\varepsilon$-convexity of f for any $c>0$.

Briefly, in [5] the following result was proved:

1. The function $fx) := \ln x$, $(x>0)$ satisfies (ii)* with $\varepsilon = 1$ but it is not c-convex for any $c>0$.

2. For all $n\in\mathbb{N}$ there exists a function f_n which satisfies (iii)* with $\varepsilon=1$ but not c-convex for any $0<c<n$.

Related to another version of approximate convexity that was studied in [2] we have

Fact 3
Assume that f *is* $(\varepsilon,1)$*-Jensen-convex in the following sense*

(i)**

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + \varepsilon|x-y| \qquad (x,y\in I,\, t\in[0,1]).$$

Then

(ii)**

$$\frac{1}{y-x}\int_x^y f(t)dt \le \frac{f(x)+f(y)}{2} + \varepsilon|x-y| \qquad (x,y\in I,\, x<y);$$

(iii)**

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y f(t)dt + \frac{\varepsilon}{2}|x-y| \qquad (x,y\in I,\, x<y).$$

Conversely, if (ii)** *and* (iii)** *hold then* f *is* $\left(\frac{3}{2}\varepsilon,1\right)$*-Jensen-convex.*

Proof. Assume that f is $(\varepsilon,1)$-convex. Then, by the main result of [2],

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)+2\varepsilon T(t)|x-y| \quad (x,y\in I,\, t\in[0,1]),$$

where $T:\mathbb{R} \to \mathbb{R}$ denotes the Takagi-function defined by

$$T(t) := \sum_{n=0}^{\infty} \frac{\mathrm{dist}(2^n t, \mathbb{Z})}{2^n} \qquad (t \in \mathbb{R}).$$

Now, integrating this inequality with respect to t over $[0,1]$, using that $\int_0^1 T(t)dt = \frac{1}{2}$, one obtains (ii)**. To deduce (iii)**, observe that (i)** implies, for $x, y \in I$, $t \in [0,1]$,

$$f\left(\frac{x+y}{2}\right) \le \frac{f(tx+(1-t)y) + f(ty+(1-t)x)}{2} + \varepsilon\,|1-2t|\,|x-y|.$$

Integrating this inequality with respect to t over $[0,1]$, one gets (iii)*.

If (ii)** and (iii)** hold then

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + \frac{3}{2}\varepsilon|x-y| \qquad (x, y \in I),$$

which means the $\left(\frac{3}{2}\varepsilon, 1\right)$-Jensen-convexity of f. □

Motivated by the above fact, we can raise the following

Problem

Does either (ii)** or (iii)** imply the $(c\varepsilon, 1)$-convexity of f for some positive constant c?

[1] M. Bessenyei, Zs. Páles, *Characterizations of convexity via Hadamard's inequality*, Math. Inequal. Appl. **9** (2006), 53–62.

[2] A. Házy, Zs. Páles, *On approximately midconvex functions*, Bull. London Math. Soc. **36** (2004), 339–350.

[3] C. T. Ng, K. Nikodem, *On approximately convex functions*, Proc. Amer. Math. Soc. **118** (1993), 103–108.

[4] C. P. Niculescu, L.-E. Persson, *Convex Functions and Their Applications. A Contemporary Approach*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 23, Springer, New York, 2006.

[5] K. Nikodem, T. Riedel, P. Sahoo, *The stability problem of the Hermite-Hadamard inequality*, submitted.

Zsolt Páles

5. Remark. *Functional equations involving weighted quasi-arithmetic means and their Gauss composition* (presented by Zs. Páles)

Let $I \subset \mathbb{R}$ be a nonvoid open interval. Let $M_i: I^2 \to I$ $(i = 1, 2, 3)$ be weighted quasi-arithmetic means with the property

$$M_3 = M_1 \otimes M_2\,,$$

where $\otimes$ denote the Gauss composition of M_1 and M_2. We consider the following two functional equations for the unknown $f: I \to \mathbb{R}$:

(1) $f(M_1(x,y)) + f(M_2(x,y)) = f(x) + f(y) \quad (x, y \in I)$,

(2) $2f(M_3(x,y)) = f(x) + f(y) \quad (x, y \in I)$.

It is known, that all solutions of (2) are solutions of (1), too. We give a complete characterization for the means M_i $(i = 1, 2, 3)$ so that arbitrary solution of (1) also satisfy (2).

Zoltán Daróczy

6. Remark.

In 1960 the following system of equalities was solved by Aczél and Gołąb (see [1], also [2])

$$H(s,t,x) = H(u,t,H(s,u,x)), \tag{1}$$

$$H(s,s,x) = x. \tag{2}$$

One can observe that equations (1) and (2) themselves do not need any algebraic structures in the domain of H so we could assume that the function H acts as follows $H: S \times S \times X \to X$ where S and X are sets.

Moreover, it is known that if $(S, +)$ is a group and $F: S \times X \to X$ satisfies the translation equation

$$F(s+t,x) = F(t,F(s,x))$$

with natural initial condition

$$F(0,x) = x, \tag{3}$$

then the function $H: S \times S \times X \to X$ defined by

$$H(s,t,x) := F(s-t,x)$$

satisfies the system of (1) and (2).

Nevertheless, condition (3) is common but in some situations is not fulfilled by solution of the translation equation. It leads to the idea of solving equation (1) without equality (2). In this direction we have proved the following proposition.

Proposition

Let S, X be sets and let $H: S \times S \times X \to X$ be a solution of equation (1). Therefore there are functions $\Phi, \Psi: S \times X \to X$ such that

$$H(s,t,x) := \Psi(t, \Phi(s,x)) \qquad \text{for every } s,t \in S,\ x \in X. \tag{4}$$

Moreover, if $\Phi, \Psi: S \times X \to X$ are functions such that for every $u \in S$: $\Psi(u,\cdot)^{-1} = \Phi(u,\cdot)$ on the set $\Phi(S \times X)$, then the function $H: S \times S \times X \to X$ given by the formula (4) is a solution of equation (1).

[1] J. Aczél, S. Gołąb, *Funktionalgleichungen der Theorie der Geometrischen Objekte*, PWN, Warszawa, 1960.

[2] Z. Moszner, *Les equations et les inégalités liées à l'équation de translation*, Opuscula Math. **19** (1999), 19–43.

Grzegorz Guzik

7. Problem.

Is the following conjecture true?

Conjecture

Let the diffeomorphism $\Psi: (0,\infty) \to (0,\infty)$ have no fixed point. If for every increasing self-diffeomorphism g of the closed interval $[0,\infty)$ the function

$$g_\Psi(x) := \Psi^{-1}\left(g\left(\Psi(x)\right)\right), \qquad x > 0,$$

(with value 0 at zero) is again a self-diffeomorphism of $[0,\infty)$*, then the derivative* $D\Psi$ *of* Ψ *is slowly varying at zero.*

For making the problem more readable, let us sketch a proof of the inverse claim. For, let the diffeomorphism Ψ have slowly varying derivative, i.e., let

$$\lim_{x\to 0}\frac{D\Psi(\lambda\cdot x)}{D\Psi(x)}=1 \qquad \text{for all } \lambda>0.$$

Then both, Ψ and Ψ^{-1} are regularly varying with exponent 1 (we are omitting the details). Moreover for the derivative of g_Ψ we have

$$Dg_\Psi(x)=\frac{D\Psi(x)}{D\Psi(\Psi^{-1}\circ g\circ\Psi(x))}\cdot Dg(\Psi(x)).$$

With the use of $Dg(0)>0$, by the regular variability of Ψ^{-1} we obtain that the ratio of the arguments of Ψ has a finite and positive limit as follows,

$$\lim_{x\to 0}\frac{x}{\Psi^{-1}\circ g\circ\Psi(x)}=\lim_{x\to 0}\frac{\Psi^{-1}\circ\Psi(x)}{\Psi^{-1}\circ g\circ\Psi(x)}=\lim_{x\to 0}\frac{\Psi(x)}{g\circ\Psi(x)}=\lim_{y\to 0}\frac{y}{g(y)}=(Dg(0))^{-1}\in(0,\infty).$$

By the slow variability of $D\Psi$ and by continuity of Dg, the limit $Dg_\Psi(0^+)$ equals $1\cdot Dg(0)$. By similar arguments from $Dg_\Psi(0)=\lim_{x\to 0}\frac{g_\Psi(x)}{x}$ one can get that $Dg_\Psi(0)=Dg(0)$, too. Thus, Dg_Ψ is continuous at zero, which closes the most important step for the inverse claim.

Joachim Domsta

8. Remark.

The Theorem formulated on p. 159 of the report on the 10th ICFEI (Ann. Acad. Paed. Cracov. Studia Math., **5** (2006)) is not true. A counterexample: $F(x,y)=y$, $(x,y)\in\mathbb{R}^2$, was communicated to the speaker by Professor Karol Baron.

The correct formulation of the result, obtained jointly with Z. Powązka (Kraków) is the following.

Theorem

Assume that $I\subset\mathbb{R}$ *is an open nonempty interval and that* (I,F) *is a group with the unit* e*. If* $\psi\colon\mathbb{R}\to\mathbb{R}$, $\psi(x+y)\le F(\psi(x),\psi(y))$, $(x,y)\in\mathbb{R}^2$, $\psi(0)=e$*, and there is a function* $\varphi\colon\mathbb{R}\to\mathbb{R}$*, such that* $\varphi(x+y)=F(\varphi(x),\varphi(y))$, $(x,y)\in\mathbb{R}^2$*, and* $\psi(x)\le\varphi(x)$, $x\in\mathbb{R}$*, then* $\psi=\varphi$*.*

Bogdan Choczewski

9. Remark. *Regular variability in functional equations*

This remark is related to the paper presented by professor Zsolt Páles (see Abstracts of Talks). Some of the results use the regular variability almost everywhere for obtaining uniqueness of the generating function from the mean, dependent additionally on some generating measure (mean of a mixed type). We want to point at the fact that the regular variability has been used already in the following (obviously much simpler) problem of restoring f from the mean defined as follows

$$M_f(x,y):=f^{-1}\left(\frac{xf(x)+yf(y)}{x+y}\right),\qquad x,y\in I \tag{1}$$

where f is a continuous and strictly monotonic function defined on an interval I of positive reals. For a point $x_0\in I$ let the auxiliary function

$$\delta_0(u):=f(x_0+u)-f(x_0)\qquad \text{whenever } x_0+u\in I \tag{2}$$

be regularly varying at 0 with non-zero exponent, i.e., let

$$\lim_{u\to 0}\frac{\delta_0(\lambda\cdot u)}{\delta_0(u)}=\lambda^\rho \qquad \text{for } \lambda>0, \text{ where } \rho\in(0,\infty). \tag{3}$$

(For measurable functions the definition is equivalent to the notion introduced by J. Karamata in [5]; for review of the regular variability see [1], [4] or [6], and for the facts suitable for the functional equations, see [1].) In terms of

$$\mu_0(u):=M_f(x_0,x_0+u)-x_0, \qquad w_0(u):=\frac{x_0+u}{2x_0+u}, \tag{4}$$

definition (1) implies the following homogeneous equation

$$\delta_0(\mu_0(u)) = w_0(u)\cdot\delta_0(u) \quad \text{whenever } u\in I-x_0, \tag{5}$$

where $I-x_0:=\{x-x_0 \,:\, x\in I\}$. It is shown in [3] that

$$f(x)=f(x_0)+(f(x_1)-f(x_0))\cdot\lim_{n\to\infty}\left(\frac{{\mu_0}^n(x-x_0)}{{\mu_0}^n(x_1-x_0)}\right)^\rho\cdot\frac{W_{0;n}(x_1-x_0)}{W_{0;n}(x-x_0)}$$

where μ_0 is given by the x_0-cut of M_f according to (4), and

$$W_{0;n}(u):=\prod_{j=0}^{n-1} w_0({\mu_0}^j(u)) \qquad \text{for } u\in I-x_0. \tag{6}$$

whenever $(x-x_0)\cdot(x_1-x_0)>0$, $x_1,x\in I$.

[1] N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and Its Applications **27**, Cambridge University Press, Cambridge – New York – New Rochelle – Melbourne – Sydney, 1987.

[2] J. Domsta, *Regularly Varying Solutions of Functional Equations in a Single Variable – Applications to the Regular Iteration*, Uniwersytet Gdański, Gdańsk, 2002.

[3] J. Domsta, J. Matkowski, *Invariance of the arithmetic mean with respect to special mean-type mappings*, Aequationes Math. **71** (2006), 70 – 85.

[4] W. Feller, *An Introduction to Probability Theory and its Applications*, vol. **2**, John Wiley and Sons, Inc., New York, 1966.

[5] J. Karamata, *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj), **4** (1930), 38–53.

[6] E. Seneta, *Regularly varying functions*, Lecture Notes in Math. **508**, Springer-Verlag, Berlin – Heidelberg – New York, 1976.

Joachim Domsta

10. Remark. *Embedding commuting functions into a regular iteration group*

An increasing continuous self-mapping $f\colon(0,\infty)\to(0,\infty)$ is said to be Szekeresian if $f(x)<x$, for all $x>0$, possesses the derivative at zero $D_0f:=\lim_{x\to 0}\frac{f(x)}{x}$ in $(0,1)$ and if the Szekeres principal function

$$\varphi_f(x\|y):=\lim_{n\to\infty}\frac{f^n(x)}{f^n(y)}, \qquad x>0,$$

is continuous for some $y > 0$. If additionally f is homeomorphic onto $(0, \infty)$ then $\varphi_f(\cdot \,\|y)$ is the unique regularly varying at zero solution of the canonical Schröder equation

$$\varphi_f(f(x)\|y) = d \cdot \varphi_f(x\|y), \qquad x > 0, \text{ where } d = D_0 f$$

equal 1 at y, for arbitrary positive y (for details, see [1]). The following are considerations which were suggested to me by prof. J. Matkowski. Let f and g be commuting Szekeresian homeomorphisms. Then $\varphi_f := \varphi_f(x\|y)$, with fixed y, satisfies

$$d \cdot (\varphi_f \circ g) = \varphi_f \circ f \circ g = (\varphi_f \circ g) \circ f,$$

which means that $\varphi_f \circ g$ is again a regularly varying solution of the canonical Schröder equation for f. By a suitable uniqueness theorem, for some positive constant C

$$\varphi_f \circ g = C \cdot \varphi_f.$$

All the facts together show, that φ_f is the Szekeres principal function for g and that $C = D_0 g$. Let us introduce $\rho := \frac{\log D_0 g}{\log D_0 f}$.

Corollary
If φ_f is homeomorphic, then there is exactly one regular iteration group containing f and g. Moreover the iterates are given by the formula:

$$f_t = (\varphi_f)^{-1} \circ (d^t \cdot \varphi_f), \quad t \in (-\infty, \infty),$$

and $f = f_1$ and $g = f_\rho$.

Conjecture
If $\rho \notin \mathbb{Q}$, then there is a regular iteration group $(f_t;\, t \in T)$ indexed by the (dense) additive subgroup T generated by 1 and ρ and such that $f = f_1$ and $g = f_\rho$.

[1] J. Domsta, *Regularly Varying Solutions of Functional Equations in a Single Variable – Applications to the Regular Iteration*, Uniwersytet Gdański, Gdańsk, 2002.

Joachim Domsta