Folia Studia Mathematica

Report of Meeting

8th International Conference on Functional Equations and Inequalities, Z lockie, September 10-15, 2001

The Eighth International Conference on Functional Equations and Inequalities, in the series of those organized by the Institute of Mathematics of the Pedagogical University in Kraków since 1984, was held from September 10 to September 15, 2001, in the hotel "Geovita" at Złockie. Circumstances independent of the organizers of both meetings caused that the Conference took place only three weeks later than the annual, 39th Symposium on Functional Equations held at Sandbjerg (Denmark, August 12-18, 2001).

A support of the Polish State Committee for Scientific Research (KBN) is acknowledged with gratitude.

The Conference was opened by the address of Prof. Dr. Michał Śliwa, the Rector Magnificus of the Pedagogical University in Kraków, who greeted the participants, thanked the organizers and wished a fruitful and nice stay in this beatiful region of Poland.

There were 56 participants who came from 8 countries: Austria (1), Germany (3), France (1), Hungary (8), Slovenia (1), The U.S.A. (1), Yugoslavia (1), and from Poland: Bielsko-Biała (1), Gdańsk (1), Gliwice (1), Katowice (11), Kraków (22), Rzeszów (3), Warszawa (1).

During 18 sessions 47 talks were delivered, mainly on functional equations in several variables and their stability (also for conditional equations), functional equations steming from independence of random matrices, special Banach space operators, iterative functional equations, iteration theory, equations and inclusions for multivalued functions and on convex functions. The following particular topics were also dealt with: generalized derivatives, ODEs in metric spaces and PDEs solved via recurrences, functional equations on quasigroups, hypergroups and other general structures, geometry of matrices, iterative functional inequalities. There were 8 contributions to problems-and-remarks sessions.

The organizing Committee was chaired by Professors Dobiesław Brydak and Bogdan Choczewski. Dr. Jacek Chmieliński acted as a scientific secretary. Miss Ewa Dudek, Miss Janina Wiercioch and Mr Władysław Wilk (technical assistant) worked in the course of preparation of the meeting and in the Conference office at Złockie.

At the same week the Annual Meeting of the Polish Mathematical Society was held in Nowy Sącz. For the first time in the history of these meetings functional equations were included in the programme. Professor Roman Ger gave an invited lecture entitled *Równania funkcyjne — zarys rozwoju i aktualny stan badań* on Wednesday, September 12. For this reason the afternoon session of 8th ICFEI were canceled.

Deeply moved by the tragic events in the United States of America, the participants expressed their sympathy and solidarity with the American nation by means of forming a hand-to-hand Solidarity Chain after Professor Thomas Riedel's (Louisville, KY) talk at Wednesday, September 12 and by suspending the session at noon on Friday, September 14.

The Conference was closed by Professor Dobiesław Brydak. The 9th ICFEI is planned to be held in September, 2003.

The Chairmen want to cordially thank the participants for their coming, presenting valuable contributions and creating the unique atmosphere of friendship and solidarity. They express the best thanks to the members of the whole office staff at Złockie for their effective and dedicated work and helpful assistance, and to the managers of the hotel "Geovita" for their hospitality and quality of services.

The abstracts of talks are printed in the alphabetical order, and the problems and remarks presented chronologically. The careful and efficient work of Dr. J. Chmieliński on completing the material and preparing (together with Mr W. Wilk) the present report for printing is acknowledged with thanks.

Bogdan Choczewski

Abstracts of Talks

Roman Badora *On almost periodic spherical functions*

We consider the functional equation

$$M_k(f(x+ky)) = g(x)h(y), \quad x, y \in G,$$

where G is a topological Abelian group, K a subgroup of its automorphisms, M stands for the invariant mean on the space of almost periodic functions defined on K and $f, g, h : G \to \mathbb{C}$ are unknown functions. We show that almost periodic solutions of this equation can be expressed by means of characters of the group G. Our considerations are motivated by the result of H. Stetkær on continuous solutions of the equation

$$\int_K f(x+ky)\,d\mu(k) = g(x)h(y), \quad x, y \in G,$$

where G and K are compact.

Anna Bahyrycz *On the indicator plurality function*

Joint work with Zenon Moszner.

A solution of the conditional functional equation:

$$f(x)\cdot f(y)\neq \underline{0} \Longrightarrow f(x+y)=f(x)\cdot f(y),$$

for which there exists a number $r\in\mathbb{R}(1)\setminus\{1\}$ such that:

$$f(rx)=f(x),$$

where $f:\mathbb{R}(n):=[0,\infty)^n\setminus\{\underline{0}\}\to\mathbb{R}(n)$, $\underline{0}:=(0,\ldots,0)\in\mathbb{R}^n$ and $x+y:=(x_1+y_1,\ldots,x_n+y_n)$, $x\cdot y:=(x_1y_1,\ldots,x_ny_n)$, $rx:=(rx_1,\ldots,rx_n)$, for $x=(x_1,\ldots,x_n)\in\mathbb{R}(n)$, $y=(y_1,\ldots,y_n)\in\mathbb{R}(n)$, is called an *indicator plurality function.*

We show that this function f, if r is a transcedental number, must have its values in the set $0(n):=\{0,1\}^n\setminus\{\underline{0}\}$ if $n\leqslant 2$, and can have the values off the set $0(n)$ if $n>2$. The problem is due to Z. Moszner.

We study under which assumptions the indicator plurality function must have its values in the set $0(n)$.

Karol Baron *Orthogonality and additivity modulo a discrete subgroup*

Let E be a real inner product space of dimension at least 2, G a topological Abelian group, and K a discrete subgroup of G. Following [1] and [2] we consider functions $f:E\to G$ continuous at least at one point and such that

$$f(x+y)-f(x)-f(y)\in K \quad \text{for all orthogonal } x,y\in E.$$

It turns out that (with no additional assumption) there exist continuous additive functions $a:\mathbb{R}\to G$ and $A:E\to G$ such that

$$f(x)-a\left(\|x\|^2\right)-A(x)\in K \quad \text{for every } x\in E.$$

[1] K. Baron and J. Rätz, *Orthogonality and additivity modulo a subgroup*, Aequationes Math. **46**(1993), 11-18.

[2] J. Brzdęk, *On orthogonally exponential and orthogonally additive mappings*, Proc. Amer. Math. Soc. **125**(1997), 2127-2132.

Lech Bartłomiejczyk *A characterization of operations invariant under bijections*

Joint work with Józef Drewniak.

We characterize n-ary operations $F : I^n \to I$ on the unit interval such that

$$F(\varphi(x_1), \dots, \varphi(x_n)) = \varphi(F(x_1, \dots, x_n))$$

holds for every increasing bijection $\varphi : I \to I$ and for every $(x_1, \dots, x_n) \in I^n$.

Bogdan Batko *On the stability of alternative Cauchy equations*

Let us consider a conditional (alternative) Cauchy equation

$$(x, y) \in \mathcal{Z} \implies f(x + y) = f(x) + f(y),$$

where $\mathcal{Z}$ is a set. We deal with the stability of conditional Cauchy equations with the condition (the set $\mathcal{Z}$) dependent on the unknown function f. Our main stability results concern Mikusiński's equation

$$f(x + y) \neq 0 \implies f(x + y) = f(x) + f(y),$$

as the fundamental example of this kind of equations.

Zoltán Boros *$\mathbb{Q}$-derivatives of Jensen-convex functions*

Definitions and properties of radial $\mathbb{Q}$-derivatives of Jensen-convex functions are presented. Some of these properties provide a characterization of Jensen-convex functions among real valued functions defined on an open interval.

Nicole Brillouet–Belluot *The ACP method for solving some composite functional equations*

We show how important for solving some composite functional equations is the ACP method based on the well-known theorem of J. Aczél (given also by R. Craigen and Z. Páles) which gives the representation of a continuous cancellative associative operation on a real interval. In particular, we find with this method all continuous solutions of the following functional equation

$$f(af(x)f(y) + b(f(x)y + f(y)x) + cxy) = f(x)f(y) \quad (x, y \in \mathbb{R})$$

which generalizes both Ebanks functional equation and Baxter functional equation.

Dobiesław Brydak *On a linear functional inequality*

Bogdan Choczewski *On a functional equation of Wilson type*

Joint work with Zbigniew Powązka.

The study is motivated by E. Wachnicki's paper [1] dealing with an integral mean value thoerem. The equation reads

$$af(x) + bf(y) = f(ax + by)\, g(y - x), \quad x, y \in X, \tag{1}$$

where a, b are some positive reals, $f : X \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ are unknown functions and X is either the real line or positive or negative half-line.

We find solutions (f, g) of equation (1) on X mainly in the case where f is locally integrable and g is continuous at the origin. In particular, among solutions exponential funtions show up. As a tool we use, among others, simple Schröder equations

$$f(px) = q\,f(x),$$

with suitable constants p and q.

[1] E. Wachnicki, *Sur un développement de la valeur moyenne*, Rocznik Nauk.-Dydakt. Wyż. Szk. Ped. Kraków. Prace Mat. **14**(1997), 35-48.

Jacek Chudziak *Functional equation of the Gołąb-Schinzel type*

We deal with the equation

$$f(x\varphi(f(y)) + y\psi(f(x))) = f(x)f(y), \qquad x, y \in \mathbb{R}, \tag{$*$}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ are unknown functions.

The equation $(*)$ is a generalization of the well known Gołąb-Schinzel equation.

Krzysztof Ciepliński *General construction of non-dense disjoint iteration groups on the circle*

Let $\mathcal{F} = \{F^v : S^1 \to S^1,\ v \in V\}$ be a disjoint iteration group on the unit circle S^1, that is a family of homeomorphisms such that

$$F^{v_1} \circ F^{v_2} = F^{v_1+v_2}, \quad v_1, v_2 \in V$$

and each F^v either is the identity mapping or has no fixed point ($(V, +)$ is a 2-divisible nontrivial Abelian group). Denote by $L_{\mathcal{F}}$ the set of all cluster points of $\{F^v(z),\ v \in V\}$ for $z \in S^1$. We give a general construction of disjoint iteration groups for which $\emptyset \neq L_{\mathcal{F}} \neq S^1$.

Stefan Czerwik *On the stability of the quadratic functional equation in some special function spaces*

Joint work with Krzysztof Dłutek.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called the quadratic functional equation. Some aspects of the stability problems for this equation in some function spaces will be discussed.

Joachim Domsta *On f-slow variability of solutions to linear equations*

Let us assume that

$$f : X \to X, \quad g : X \to \mathbb{R}_+ := (0, \infty), \quad \text{where } X \neq \emptyset. \tag{1}$$

Definition

We say, that function $\Psi : X \to \mathbb{R}_+$ is slowly varying on the orbits of f (briefly: f-slowly varying), whenever for every pair $(x, y) \in X^2$ the limit

$$\Psi_f(x|y) := \lim_{n\to\infty} \frac{\Psi(f^n(x))}{\Psi(f^n(y))}$$

equals 1.

The main results concerns the solutions of the following equation

$$\Psi(f(x)) = g(x) \cdot \Psi(x), \quad x \in X. \qquad \mathbf{Eqn}[f, g]$$

If $\Psi(y) \neq 0$ then we have the following system of equations implied by $\mathbf{Eqn}[f, g]$,

$$\frac{\Psi(x)}{\Psi(y)} = \frac{\Psi\left(f^n(x)\right)}{\Psi\left(f^n(y)\right)} \cdot \gamma_{f,g;n}(x|y), \quad \text{for } n \in \mathbb{N}, \tag{2}$$

where

$$\gamma_{f,g;n}(x|y) := \prod_{j=0}^{n-1} \frac{g(f^j(y))}{g(f^j(x))}, \quad \text{for } x, y \in X. \tag{3}$$

Therefore, the limit $\Psi_f(x|y)$ exists in $\mathbb{R} \setminus \{0\}$, iff the limit

$$\gamma_{f,g}(x|y) := \lim_{n\to\infty} \gamma_{f,g;n}(x|y)$$

exists in $\mathbb{R} \setminus \{0\}$ and $\Psi(x) \cdot \Psi(y) \neq 0$. If this holds, then

$$\frac{\Psi(x)}{\Psi(y)} = \Psi_f(x|y) \cdot \gamma_{f,g}(x|y).$$

Theorem

The equation $\mathbf{Eqn}[f, g]$ *possesses an f-slowly varying solution if and only if the principial function $\gamma_{f,g}(\cdot|y)$ exists in the class of positive functions on X for some/every $y \in X$ and $\lim_{n\to\infty} g\left(f^n(x)\right) = 1$ for all $x \in X$. Then all f-slowly varying solutions are of constant non-zero sign and they are represented as follows,*

$$\Psi(x) = \Psi(y) \cdot \gamma_{f,g}(x|y), \quad \textit{for all } x, y \in X. \tag{4}$$

Corollary

Function $\Pi(x) := \Gamma(x+1)$, $x > 0$, *is the unique solution of the problem*

$$\Pi(x+1) = (x+1) \cdot \Pi(x), \quad \text{for } x > 0, \text{ and } \Pi(1) = 1. \tag{5}$$

for which $\tilde{\Pi}(x) := e^{x-1} \cdot x^{-x} \cdot \Pi(x)$, $x > 0$, *is* $(x+1)$*-slowly varying.*

Tibor Farkas *On functions additive with respect to algorithms*

We prove that for an arbitrary interval filling sequence there exist two algorithms such that the additivity of a function with respect to them implies its linearity. In contrast to some known results, we prove the linearity of the function without requiring any special properties for the interval filling sequence and any regularity properties for the function.

Roman Ger *Rational associative operations and the corresponding addition formulas*

Joint work with Katarzyna Domańska.

Our goal is to present a method of solving Cauchy type functional equations of the form

$$f(x+y) = F(f(x), f(y))$$

(addition formulas) where the given binary operation F defined on a subset of the real plane $\mathbb{R}^2$ is rational and associative.

The key tools we are using are two representation theorems for rational associative mappings on the plane due to A. Chéritat (1999) and some results on Cauchy type equations assumed almost everywhere.

The solutions are found in the class of functions with "small" counterimages of singletons (i.e. belonging to a proper linearly invariant set ideal in the domain space).

Attila Gilányi *On a uniqueness problem for homogeneous utility representations*

Joint work with C. T. Ng.

We investigate a uniqueness question, posed in [1], for homogeneous utility representations. In order to answer it, we solve the functional equation

$$F_1(t) - F_1(t+s) = F_2[F_3(t) + F_4(s)]$$

under slightly different conditions as it was done in [4] (cf. also [5], [2] and [3]).

[1] J. Aczél, R. D. Luce, C. T. Ng, *Separability, segregation, and homogeneity in a theory of utility*, manuscript.

[2] J. Aczél, R. Ger, A. Járai, *Solution of a functional equation arising from utility that is both separable and additive*, Proc. Amer. Math. Soc. **127**(1999), 2923-2929.

[3] J. Aczél, Gy. Maksa, Zs. Páles, *Solution of a functional equation arising in an axiomatization of the utility of binary gambles*, Proc. Amer. Math. Soc. **129**(2001), 483-493.

[4] J. Aczél, Gy. Maksa, C. T. Ng, Zs. Páles, *A functional equation arising from ranked additive and separable utility*, Proc. Amer. Math. Soc. **129**(2001), 989-998.

[5] A. Lundberg, *On the functional equation* $f(\lambda(x)+g(y)) = \mu(x)+h(x+y)$, Aequationes Math. **16**(1977), 21-30.

Máté Györy *Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving orthogonality*

Attila Házy *On approximately Jensen-convex functions*

Joint work with Zsolt Páles.

A function $f : D \to \mathbb{R}$ is called (ε, δ)-midconvex if

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2}\left(f(x)+f(y)\right) + \delta + \varepsilon \mid x-y \mid .$$

Our main result shows that if f is locally bounded from above and (ε, δ)-midconvex, then f satisfies the following convexity inequality

$$f\left(\lambda x + (1-\lambda)y\right) \leqslant \lambda f(x) + (1-\lambda) f(y) + 2\delta + 2\varepsilon\varphi(\lambda) \mid x-y \mid$$

for every $x, y \in D$ and $\lambda \in [0,1]$, where φ is defined by

$$\varphi(\lambda) = \begin{cases} -2\lambda \log_2 \lambda, & 0 \leqslant \lambda \leqslant \frac{1}{2}, \\ -2(1-\lambda)\log_2(1-\lambda), & \frac{1}{2} \leqslant \lambda \leqslant 1. \end{cases}$$

The case $\varepsilon = 0$ of the result reduces to that of Nikodem and Ng from 1993.

Witold Jarczyk *Reversibility on the circle*

A homeomorphism of a topological space is said to be *continuously reversible* if it is a composition of two continuous involutions or, equivalently, the homeomorphism and its inverse function are conjugated by a continuous involution. In the talk continuous reversibility of homeomorphisms of the circle is studied, especially those with no periodic points. This is a continuation of a research made by the author for the real case.

Hans-Heinrich Kairies *Images and pre-images of a Banach space operator*

The operator F, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x)$$

is a continuous automorphism on the Banach space B of real bounded functions and on several of its closed subspaces. F is known to generate continuous nowhere differentiable [cnd] functions from simple elements of B (Weierstrass, Takagi). But neither the image $F[\mathcal{N}]$ nor the pre-image $F^{-1}[\mathcal{N}]$ of the set $\mathcal{N}$ of cnd functions from B is know at present. We describe some subsets of $F[\mathcal{N}]$ and of $F^{-1}[\mathcal{N}]$ as well as images and pre-images of other function sets.

Barbara Koclęga-Kulpa *On a conditional functional equation in normed spaces*

We deal with the functional equation

$$\|f(x \cdot y)\| = \|f(x) + f(y)\| \tag{1}$$

considered by Roman Ger for function f mapping a given group into a strictly convex space (see [1]). We consider equation (1) for function f defined on the algebra $M_n(\mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ of all real or complex $n \times n$ matrices. Then equation (1) looks as follows

$$A \cdot B \neq [0] \Longrightarrow \|f(A \cdot B)\| = \|f(A) + f(B)\|, \tag{2}$$

where $A, B \in M_n(\mathbb{K})$ and $[0]$ stands for a zero matrix.

[1] R. Ger, *On a characterization of strictly convex spaces*, Atti Accad. Sci. Torino Cl. Sci Fis. Mat. Natur. **127**(1993), 131-138.

Zygfryd Kominek *On the quasisymmetry quotient*

We will consider the equation of the form

$$\frac{f(x+h) - f(x)}{f(x) - f(x-h)} = \varphi(h)\psi(x),$$

where $f, \psi : \mathbb{R} \to \mathbb{R}$, and $\varphi : (0, \infty) \to \mathbb{R}$ are unknown functions.

Aleksandar Krapež *Fully debalanced functional equations on almost quasigroups*

A quasigroup may be characterized by the property that the (left or right) translation by any element is permutation. If we allow that some translations may be constant functions, we get so called *almost quasigroups*. The quasigroups are characterized as isotopes of loops. Similarly we can characterize almost quasigroups as isotopes of either loops or loops with external zero added.

A special class of linear functional equations on almost quasigroups is considered. They are called *fully debalanced* and characterized by a single appearance of variables (both object and functional) on the left hand side of the given equation $u = b$, while the right hand side (b) is a constant.

The general solution of such functional equation is described in terms of the structure of the tree of the term u.

Zbigniew Leśniak *On the structure of equivalence classes of a relation for a free mapping*

We present some results concerning an equivalence relation for a given free mapping f of the plane. We discuss the problem if the equivalence classes of the relation are invariant under f and if the restriction of f to each of them is conjugate to a translation.

Lajos Molnár *Ortho-order automorphisms of Hilbert space effect algebras*

Joint work with Zsolt Páles.

Let H be a (real or complex) Hilbert space. The effect algebra of H is the operator-interval $[0, I]$ of all positive (selfadjoint, bounded linear) operators on H which are bounded by the identity I. Effect algebras play very important role in the mathematical foundations of quantum mechanics (see, for example, [1]). It is well-known that if the dimension of H is at least 3, then the $\perp$-order automorphisms of $[0, I]$ (which are the bijective transformations of the effect algebra that preserve the order $\leqslant$ in both directions and also preserve a kind of orthocomplementation $\perp$: $E \to I - E$) are implemented by unitary or antiunitary operators on H. In fact, the proof is usually based on the fundamental theorem of projective geometry which holds true only in spaces of dimension not less than 3. Because of the importance of effect algebras, it is a natural problem to clarify the situation in the 2-dimensional case. In fact, Cassinelli, De Vito, Lahti and Levrero faced this question in their paper [2]. Moreover, in their recent work [3], Lahti, Mączyński and Ylinen showed that if the considered automorphism is induced via the functional calculus by a Borel function of the interval $[0, 1]$, then it is necessarily the identity. The aim of this talk is to present the complete solution of the problem. Namely, we have the following result.

Theorem

Let H be a 2-dimensional (real or complex) Hilbert space and let $[0, I]$ be the effect algebra of H. Let $\phi : [0, I] \to [0, I]$ be a bijective transformation with the property that

$$E \leqslant F \iff \phi(E) \leqslant \phi(F) \quad \textit{and} \quad \phi(I - E) = I - \phi(E)$$

holds for every $E, F \in [0, I]$. Then there exists an either unitary or antiunitary operator U on H such that

$$\phi(E) = UEU^* \quad (E \in [0, I]).$$

[1] G. Ludwig, *Foundations of Quantum Mechanics, Vol. I*, Springer Verlag, 1983.

[2] G. Cassinelli, E. De Vito, P. Lahti and A. Lavrero, *A theorem of Ludwig revisited*, Found. Phys. **30**(2000), 1755-1761.

[3] P. Lahti, M. Mączyński, K. Ylinen, *A note on order and orthocomplementation preserving automorphisms of the set of effect operators on a Hilbert space*, Lett. Math. Phys. **55**(2001), 43-51.

[4] L. Molnár, Zs. Páles, *$\perp$-order automorphisms of Hilbert space effect algebras: The two-dimensional case*, J. Math. Phys. **42**(2001), 1907-1912.

Zenon Moszner *La fonction d'indice et la fonction exponentielle*

On donne une laison entre la fonction d'indice, c. à d. la solution de l'équation conditionnelle

$$f(c) \cdot f(d) \neq \underline{0} \Longrightarrow f(c+d) = f(c) \cdot f(d),$$

ou $f : \mathbb{R}(p) \to \mathbb{R}(p)$, $\mathbb{R}(p) = (0, +\infty)^p \setminus \{(0, \ldots, 0)\}$, $\underline{0} = (0, \ldots, 0) \in \mathbb{R}^p$, $c = (c_1, \ldots, c_p) \in \mathbb{R}(p)$, $d = (d_1, \ldots, d_p) \in \mathbb{R}(p)$, $c + d = (c_1 + d_1, \ldots, c_p + d_p)$, $c \cdot d = (c_1 d_1, \ldots, c_p d_p)$, et la fonction exponentielle, c. à d. la solution $f : \mathbb{R}(p) \to \mathbb{R}(p)$ de l'équation

$$f(c+d) = f(c) \cdot f(d).$$

Kazimierz Nikodem *On K-λ-convex set-valued functions*

Let K be a convex cone in a vector space Y, $I \subset \mathbb{R}$ be an interval and $\lambda : I^2 \to (0,1)$ be a given function.

A set-valued function $F : I \to \mathrm{n}(Y)$ is called *K-λ-convex* if

$$\lambda(x,y)F(x) + (1 - \lambda(x,y))F(y) \subset F(\lambda(x,y)x + (1 - \lambda(x,y))y) + K$$

for all $x, y \in I$ and $\in [0,1]$.

F is K-convex if

$$tF(x) + (1-t)F(y) \subset F(tx + (1-t)y) + K$$

for all $x, y \in I$ and $t \in [0,1]$.

The following set-valued generalization of a result proved recently by Zs. Páles [1] holds:

THEOREM

Let K be a closed convex cone in a real locally convex space Y, $I \subset \mathbb{R}$ be an open interval and $\lambda : I^2 \to (0,1)$ be a function continuous in each variable. If a set-valued function $F : I \to \mathrm{c}(Y)$ is K-λ-convex and locally K-upper bounded at every point, then it is K-convex.

[1] Zs. Páles, *Bernstein-Doetsch-type results for general functional inequalities*, Rocznik Nauk. Dydakt. Akad. Pedagog. w Krakowie, Prace Matematyczne **17**(2000), 197-206.

Zsolt Páles *Approximately convex functions*

A real valued function f defined on a real interval I is called (ε, δ)-convex if it satisfies

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y) + \varepsilon t(1-t)|x-y| + \delta$$

for every $x, y \in I$, $t \in [0,1]$.

The main results offer various characterizations for (ε, δ)-convexity. One of them states that f is (ε, δ)-convex for some positive ε and δ if and only if f can be decomposed into the sum of a convex function, a function with bounded supremum norm, and a function with bounded Lipschitz-modulus. In the special case $\varepsilon = 0$, the results reduce to that of Hyers, Ulam, and Green obtained in 1952 concerning the so called δ-convexity.

[1] J. W. Green, *Approximately convex functions*, Duke Math. J. **19**(1952), 499-504.

[2] D. H. Hyers, S. M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc. **3**(1952), 821-828.

[3] Zs. Páles, *On approximately convex functions, Proc. Amer. Math. Soc.*, to appear.

Tomasz Powierża *Higher order set-valued iterative roots of bijections*

Let f be a self-mapping of a non-empty set X and $r \geqslant 2$ be a positive integer. We say that a function $G : X \to 2^X$ is a *set-valued iterative root of order* r of f if

$$f(x) \in G^r(x) \quad \text{for} \quad x \in X,$$

where

$$G^0(x) := \{x\},$$

$$G^{n+1}(x) := \bigcup_{y \in G^n(x)} G(y)$$

for $x \in X$ and $n \in \mathbb{N}_0$.

We present a simple construction of a class of set-valued iterative roots of bijections. These roots have at most two-elementary values and coincide with normal iterative root if they exist. Moreover, every iterative root can be obtained using this construction.

Maria Ewa Pliś *Summability of formal solutions to Laplace type PDE's*

We consider a nonlinear differential equation

$$P(D)u = \alpha u^m, \tag{1}$$

where P is a polynomial of two variables, $m \in \mathbf{N}$, $m \geqslant 2$. We construct a formal solution

$$u(x) = T[e^{-xz}], \tag{2}$$

for some Laplace distribution T. Applying $P(D)$ to u in the form (3) we arrive at the convolution equation

$$P(Z)T = \alpha T^{*m}. \tag{3}$$

We find a solution of this equation in the form of formal series

$$T = \sum_{k=0}^{\infty} T_k$$

of Laplace distributions T_k. We show that assuming some properties of the set $CharP$ we get some Gevrey class of such solutions.

Thomas Riedel *Some results relating to Flett's mean value theorem*

We present joint work with M. Dao and P. K. Sahoo. This talk consists of 3 results, which are connected with Flett's mean value theorem and use similar methods of proof. First we show that Flett's theorem as well as the generalization by Trahan and by Riedel and Sahoo still hold for functions, which are approximately differentiable. Second, we show that for a large class of functions, the Flett mean value point is stable in the sense of Hyers and Ulam. Finally we give another mean value theorem, similar in type to Flett's, namely

If f is differentiable on $[a, b]$, then there is an $\eta \in (a, b)$ such that

$$\frac{1}{\eta - a}\left[f'(\eta) - \frac{f(\eta) - f(a)}{\eta - a}\right] + \frac{1}{\eta - b}\left[f'(\eta) - \frac{f(\eta) - f(b)}{\eta - b}\right] = \frac{f'(b) - f'(a)}{b - a}.$$

We conclude with a functional equation derived from this.

Maciej Sablik *A method of solving equations characterizing polynomial functions*

We present a lemma which generalizes some ealier results of W. H. Wilson and L. Székelyhidi and can be used as a tool in solving several equations characterizing polynomial functions, in particular some of those stemming from mean value theorems, and supposed to hold for functions defined in Abelian groups.

We also quote another lemma proved by I. Pawlikowska who has generalized both our result as well as a lemma of Z. Daróczy and Gy. Maksa thus getting a method of characterizing polynomial functions as solutions of equations assumed to hold on convex subsets of linear spaces. The talk is illustrated by a number of examples.

Adolf Schleiermacher *Some consequences of a theorem of Liouville*

Let S denote the automorphism group of the Euclidean space E of n dimensions comprising Euclidean motions as well as similarities. Let h be a C^1 diffeomorphism of E not contained in S. We are interested in the group G generated by h and S acting on E and in its invariants, i.e. real valued functions satisfying $f(g(P_0), g(P_1), \ldots, g(P_m)) = f(P_0, P_1, \ldots, P_m)$ for all $g \in G$ and $(P_0, P_1, \ldots, P_m) \in D \subseteq E^{m+1}$. Using Liouville's theorem on conformal mappings in space we shall prove: Under the action of G on E^{n+1} every orbit containing a non-degenerate n-simplex is dense in E^{n+1}. As a corollary we obtain: For $m \leqslant n$ any continuous $(m+1)$-place invariant of G whose domain of definition D contains a non-degenerate m-simplex is constant.

It may be conjectured that if h is not an affine mapping, then the group G as defined above is $(n+1)$-fold transitive. We shall show that this is in fact true for a particular class of such groups.

Stanisław Siudut *Some remarks on measurable groups*

We give an example of a complete measurable group (G, Σ, λ) such that

1. $\lambda(G) = \infty$,

2. λ is not invariant under symmetry with respect to zero.

Andrzej Smajdor *Regular set-valued iteration semigroups and a set-valued differential problem*

The connection between differentiable iteration semigroups of continuous linear set-valued functions and set-valued solutions of some linear ordinary differential equations is considered.

Wilhelmina Smajdor *Entire solutions of a functional equation of Pexider type*

All entire solutions of the functional equation

$$|f(x+y)| + |g(x-y)| = |h(x+\overline{y})| + |k(x-\overline{y})|$$

are determined.

Paweł Solarz *On iterative roots of a homeomorphism of the circle with an irrational rotation number*

Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle with the positive orientation, $F : S^1 \to S^1$ be a homeomorphism with an irrational rotation number $\alpha(F)$. Let $L_F(z)$ be the set of all cluster points of $\{F^n(z),\ n \in \mathbb{Z}\}$ for $z \in S^1$. Let $S^1 \setminus L_F \neq \emptyset$. Suppose that the iterative kernel of F, $K_F := \varphi[S^1 \setminus L_F]$, has the property $(\sqrt[n]{s})_m K_F = K_F$, where $(\sqrt[n]{s})_m = e^{2\pi i \frac{1}{n}(\alpha(F)+m)}$ for $m \in \{0, \ldots, n-1\}$, $n \in \mathbb{N}$. In this case F has infinitely many iterative roots, i.e., continuous solution of the equation $G^n = F$ depends on an arbitrary function.

Joanna Szczawińska *On Lipschitz midconcave multifunctions*

We give a version of the Bernstein-Doetsch Theorem for multifunctions with bounded and convex values. If a J-concave set-valued function is lower bounded on a ball then it is locally Lipschitzian.

László Székelyhidi *Exponential polynomials on polynomial hypergroups*

Joint work with Ágota Orosz.

Hypergroups have been investigated since the pioneer works of C. F. Dunkl, R. I. Jewett and R. Spector. The presence of translation operators makes it possible to investigate some classical functional equations on hypergroups. Here we present results concerning additive, exponential and polynomial functions on polynomial hypergroups.

Tomasz Szostok *On some conditional functional equations*

We are looking for an unconditional functional equation which would preserve the properties of a given conditional equation. We consider the well known orthogonal Cauchy equation

$$x \perp y \Longrightarrow f(x+y) = f(x) + f(y)$$

and the Ptolemaic equation

$$x \perp y \Longrightarrow f(x)^2 + f(y)^2 = f(x+y)f(x-y)$$

which was recently considered by Margherita Fochi. For both of these equations such unconditional equations are found.

Peter Šemrl *Geometry of matrices*

Two matrices are said to be *adjacent* if the rank of their difference is one. Hua's fundamental theorems of geometry of matrices characterize bijective maps preserving adjacency in both directions on various spaces of matrices. We present some recent results on such maps.

Jacek Tabor *Differential equations in metric spaces*

We generalize the notion of an autonomous differential equation to a complete metric space X. Instead of the vector field we assume that we are given a function $F : X \to C(\mathbb{R}_+, X)$ (the space of continuous functions from $\mathbb{R}_+$ to X such that $F(x)[0] = x$ for every $x \in X$).

We say that a continuous function $u : [0, T) \to X$ is a solution to the differential equation

$$du/dt = F(u)$$

if it is locally tangent to F, that is if

$$\lim_{h \to 0^+} \frac{d(u(t+h), F(u(t))[h])}{h} = 0 \quad \text{for } t \in [0, T).$$

Under some natural assumption on F we prove analogues of the classical Peano and Picard existence results.

Józef Tabor *On the Hyers operator*

Joint work with Jacek Tabor.

Let X be a metric space, and let $K \subset X$. Assume that for every $x \in X$ there exists a unique best approximation from K denoted by $\Pi_k(x)$. Properties of the mapping Π_k are investigated. As a corollary we obtain a partial solution to the problem of Z. Moszner concerning continuity of the Hyers operator.

Peter Volkmann *A characterization of the Dinghas derivative*

The generalized n-th derivative

$$D^n f(\alpha) = \lim_{\substack{x \leqslant \alpha \leqslant y \\ y - x \downarrow 0}} \left(\frac{n}{y-x} \right)^n \Delta^n_{\frac{y-x}{n}} f(x)$$

(Alexander Dinghas 1966) of $f : \mathbb{R} \to \mathbb{R}$ at $\alpha \in \mathbb{R}$ is characterized by

$$f(\alpha + t) = g(t) + D^n f(\alpha) \cdot \frac{t^n}{n!} + o(t^n) \quad (t \to 0),$$

where g is a polynomial function of degree at most $n-1$, i.e. $\Delta^n_x g(y) = 0$ $(x, y \in \mathbb{R})$. For continuous f this result is known from my doctoral thesis at Freie Universität Berlin 1971.

Anna Wach-Michalik *Convexity and functional equations connected with Euler's Beta function*

The function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula $\beta(x) = B(x, x)$ (where B is Euler's Beta function) is a particular solution of the functional equation:

$$f(x+1) = \frac{x}{2(2x+1)} f(x) \quad \text{and} \quad f(1) = 1. \tag{1}$$

We have proved a theorem, similar to that due to H. Bohr and J. Mollerup.

THEOREM

If $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is a solution of* (1), log-*convex on* $(\gamma, +\infty)$ *for some* $\gamma \geqslant 0$, *then* $f = \beta$.

We also examine the set

$$M_\beta = \{g : \mathbb{R}_+ \to \mathbb{R} : \text{for every } f \text{ satysfying (1) if } g \circ f \text{ is convex then } f = \beta\}.$$

Janusz Walorski *On the existence of continuous and smooth solutions of the Schröder equation*

We consider continuous and smooth solutions of the Schröder equation

$$\varphi(f(x)) = A\varphi(x),$$

and the linear equation

$$\varphi(f(x)) = g(x)\varphi(x) + F(x)$$

using the method motivated by [1] and [2].

[1] G. Belitskii, V. Tkachenko, *On solvability of linear difference equations in smooth and real analytic vector functions of several variables*, Integr. Equat. Oper. Th. **18**(1994), 123-129.

[2] J. Morawiec, J. Walorski, *On the existence of smooth solutions of linear functional equations*, Integr. Equat. Oper. Th. **39**(2001), 222-228.

Jacek Wesołowski *Functional equations related to independence properties of random matrices*

Problems of characterizations of probability measures can often be reduced to functional equations. For instance the celebrated Lukacs [5] theorem on characterizing the gamma distribution by independence of the sum and the quotient of two independent non-degenerate positive random variables, if existence of densities is assumed, reduces to the problem of solution of the following equation:

$$f_1(u)f_2(v) = vf_3(uv)f_4((1-u)v), \quad u \in (0,1),\ v \in (0,\infty),$$

where f_i, $i = 1,2,3,4$, are integrable non-negative functions.

Matrix variate versions of this result are concerned with, so called, Wishart distribution, which is a probability measure is the cone $\mathcal{V}_+$ of positive definite symmetric, say $n \times n$, matrices, defined by the density

$$\gamma_{p,a}(dy) = \frac{(\det a)^p}{\Gamma_n(p)}(\det y)^{p-\frac{n+1}{2}} \exp\left(-(a,y)\right) I_{\mathcal{V}_+}(y)dy, \quad y \in \mathcal{V}_+,$$

where Γ_n is the n-variate Gamma function, $a \in \mathcal{V}_+$ and $p > \frac{n-1}{2}$. Such characterizations of the Wishart distribution were studied, for instance in Casalis and Letac [2] and Letac and Massam [3] by the Laplace transform technique. Their setting was based on certain invariance property of the "quotient" of random matrices. So, in a sense, there was a strong belief that this invariance property is somehow deeply rooted in the problem under study (it is satisfied trivially in the univariate case). However the approach exploiting densities, developed in Bobecka and Wesołowski [1], reveals that no invariance property is needed in order to characterize the Wishart distribution by the Lukacs type independence property. Two basic steps in the proof are connected with solutions of the following equations:

$$a(x) = g(yxy) - g(y(e-x)y), \quad x \in \mathcal{V}_+, \quad y \in \mathcal{D} = \{z \in \mathcal{V}_+ : e - z \in \mathcal{V}_+\},$$

$$a_1(x) + a_2(y) = g(yxy) + g(y(e-x)y), \quad x \in \mathcal{V}_+, \quad y \in \mathcal{D}.$$

They are solved under some smoothness conditions imposed on functions present in the equations.

Observe that these equations look somewhat similar to the multiplicative Cauchy equation in the cone $\mathcal{V}_+$

$$f(x)f(y) = f\left(x^{\frac{1}{2}}yx^{\frac{1}{2}}\right), \quad x, y \in \mathcal{V}_+,$$

where $f : \mathcal{V}_+ \to (0, \infty)$. It appears that, at least, under differentiability assumption, the only solution of this equation has the form $f(x) = (\det x)^\lambda$, where λ is a real number.

Similar equations related to characterizations of matrix variate GIG and gamma distributions by, so called Matsumoto-Yor independence property, will be also presented. This part of the talk will follow Letac and Wesołowski [4] and Wesołowski [6].

Finally, a new independence property for matrix variate beta distributions will be shown. Here the respective characterization question leads to an open question: the functional equation for densities is still under study.

[1] K. Bobecka, J. Wesołowski, *The Lukacs-Olkin-Rubin theorem without invariance of the "quotient"*, Preprint Mar.01, 2001.

[2] M. Casalis, G. Letac, *The Lukacs-Olkin-Rubin characterization of Wishart distributions on symmetric cones*, Ann. Statist. **24**(1996), 763-786.

[3] G. Letac, H. Massam, *Quadratic and inverse regressions for Wishart distributions*, Ann. Statist. **26**(1998), 573-595.

[4] G. Letac, J. Wesołowski, *An independence property for the product of GIG and gamma laws*, Ann. Probab. **28**(2000), 1371-1383.

[5] E. Lukacs, *A characterization of the gamma distribution*, Ann. Math. Statist. **26**(1955), 319-324.

[6] J. Wesołowski, *The Matsumoto-Yor independence property for GIG and Gamma laws, revisited*, Math. Proc. Cambridge Philos. Soc. (2001), to appear.

Problems and Remarks

1. Problem. (On approximate sandwich theorems)
Let S be an abelian semigroup. Assume that $p, q : S \to \mathbb{R}$ are functions such that $q \leqslant p$ and p is ε-subadditive (i.e. $p+\varepsilon$ is subadditive), q is δ-superadditive (i.e. $q-\delta$ is superadditive) for some nonnegative ε, δ.

Then, by a well-known sandwich theorem, there exists an additive function $\psi : S \to \mathbb{R}$ such that

$$q - \delta \leqslant \psi \leqslant p + \varepsilon. \tag{1}$$

Define

$$\varphi(x) = \begin{cases} q(x) & \text{if } \psi(x) < q(x), \\ \psi(x) & \text{if } q(x) \leqslant \psi(x) \leqslant p(x), \\ p(x) & \text{if } p(x) < \psi(x). \end{cases}$$

Obviously, $q \leqslant \varphi \leqslant p$ and, due to (1), $\psi - \varepsilon \leqslant \varphi \leqslant \psi + \delta$. Thus, for $x, y \in S$, we have

$$\varphi(x+y) \leqslant \psi(x+y) + \delta \leqslant \psi(x) + \psi(y) + \delta \leqslant \varphi(x) + \varphi(y) + (2\varepsilon + \delta).$$

Similarly,

$$\varphi(x+y) \geqslant \varphi(x) + \varphi(y) - (2\delta + \varepsilon).$$

Therefore φ is $(2\varepsilon+\delta)$-subadditive and $(2\delta+\varepsilon)$-superadditive.

Does there exist a function f, $q \leqslant f \leqslant p$ such that f is ε-subadditive and δ-superadditive?

Zsolt Páles

2. Problem and Remarks. (Communicated by J. Aczél and presented by Zenon Moszner.)
Genèse: Soit x le nombre des travailleurs d'une fabrique, y le capital de roulement, $z = F(x, y)$ la production à temps t_0 et $x', y', z' = F(x', y') : \mathbb{R}_+^2 \to \mathbb{R}_+$ les mêmes valeurs à temps t. Nous supposons que $x' = \Phi(x, y, t) : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}_+$, $y' = \Psi(x, y, t) : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}_+$ et que

(a) pour chaque t_1 et t_2 il existe un t_3 tel que

$$\Phi[\Phi(x, y, t_1), \Psi(x, y, t_1), t_2] = \Phi(x, y, t_3),$$

$$\Psi[\Phi(x, y, t_1), \Psi(x, y, t_1), t_2] = \Psi(x, y, t_3),$$

(b) pour chaque t il existe $\bar{t}$ tel que

$$\Phi[\Phi(x,y,t),\Psi(x,y,t),\bar{t}\,] = x,$$

$$\Psi[\Phi(x,y,t),\Psi(x,y,t),\bar{t}\,] = y,$$

(c) il existe t_0 pour lequel

$$\Phi(x,y,t_0) = x \quad \text{et} \quad \Psi(x,y,t_0) = y.$$

Le problème est suivant: Existe-il pour chaque $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ continue, non-décroissante par rapport à chaque variable et quasi-concave des fonctions (non-banales) $\Phi, \Psi : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}_+$ continues, satisfaisantes aux conditions (a), (b), (c) et une fonction $H : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ continue telles que

$$H(F(x,y)t) = F(\Phi(x,y,t),\Psi(x,y,t))? \tag{E}$$

En interprétation économique: la production à temps t ne dépend-elle que de la production $F(x,y)$ à temps t_0 et du temps t?
Remarques.

1. Pour la fonctions (banales) $\Phi(x,y,t) = x$, $\Psi(x,y,t) = y$ et $H(z,t) = z$ l'équation (E) a lieu pour chaque $F : \mathbb{R}_+^2 \to \mathbb{R}_+$.
2. La condition (c) est une conséquence de (a) et (b) et les conditions (a) et (b) désignent que la famille $B = \{(\Phi(x,y,t),\Psi(x,y,t))\}_{t\in\mathbb{R}}$ est un groupe des bijections de $\mathbb{R}_+^2$ sur $\mathbb{R}_+^2$ par rapport à la superposition.
3. L'existence de la fonction H est équivalente à la condition:

   $$\begin{aligned}&F(x_1,y_1) = F(x_2,y_2) \Longrightarrow\\ &\forall\, t \in \mathbb{R} : F(\Phi(x_1,y_1,t),\Psi(x_1,y_1,t)) = F(\Phi(x_2,y_2,t),\Psi(x_2,y_2,t))\end{aligned}$$

   et cette implication désigne que la famille des niveaux de F (level sets) est invariante par rapport à la famille B, c.à d. chaque niveau est transformé par chaque bijection de B à un niveau (le même ou l'autre).
4. Le problème de Mitchell est done équivalent an suivant:

   – existe-il pour chaque $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ continue, non-décroissante et quasi-concave un groupe continu des bijections continues de $\mathbb{R}_+^2$ telles que la famille des niveaux de F est invariante par rapport à ces bijections.

   La réponse ne dépend done que des niveaux de F. Par exemple les orbites de B sont invariantes par rapport à B.

T. Mitchell (Southern Illinois University, Carbondale)

3. Remark.

Bogdan Choczewski

4. Problem. (On (ε, δ)-Wright-convex functions)
Let I be an open interval and let $\varepsilon, \delta \geqslant 0$. If $A : \mathbb{R} \to \mathbb{R}$ is an additive function, $h : I \to \mathbb{R}$ is bounded function with $\|h\| \leqslant \delta$, $l : I \to \mathbb{R}$ is Lipschitz with $\operatorname{Lip}(l) \leqslant \varepsilon$, $g : I \to \mathbb{R}$ is a convex function, then the function

$$f = A + g + h + l \tag{1}$$

satisfies the following inequality

$$f(tx + (1-t)y) + f((1-t)x + ty) \leqslant f(x) + f(y) + 4\delta + 4\varepsilon t(1-t)|x-y|. \tag{2}$$

If $f : I \to \mathbb{R}$ is any function that satisfies (2), then does f enjoy the decomposition (1) where A is additive, g is convex, h is bounded and l is Lipschitz? If $\varepsilon = 0$ then the affirmative answer was found by J. Mrowiec.

Zsolt Páles

5. Remark. (Solution to the Problem 1. on approximate sandwich theorems)

THEOREM

Let S be an abelian semigroup, $\varepsilon, \delta \geqslant 0$ and let $p : S \to \mathbb{R}$ be an ε-subadditive, $q : S \to \mathbb{R}$ be a δ-superadditive function such that $q \leqslant p$. Then there exists an ε-subadditive and δ-superadditive function φ such that $q \leqslant \varphi \leqslant p$.

Proof. By Zorn's lemma, we can find a minimal (with respect to the pointwise ordering) ε-subadditive function p_0 such that $q \leqslant p_0 \leqslant p$. Similarly, we can find a maximal δ-superadditive function q_0 such that $q \leqslant q_0 \leqslant p_0 \leqslant p$. We show that $q_0 = p_0$, hence $\varphi = q_0 = p_0$ will be the separating function in question.

Assume that there exists $u \in S$ such that $q_0(u) < p_0(u)$. Choose c so that $q_0(u) < c < p_0(u)$. Define $\overline{p} : S \to \mathbb{R}$ by

$$\overline{p}(t) = \inf \left\{ kc + \sum_{i=1}^{n} p_0(x_i) + (k+n-1)\varepsilon : \ k, n \geqslant 0, \ ku + \sum_{i=1}^{n} x_i = t \right\}$$

Then (with $k = 0$, $n = 1$, $x_1 = t$) we get $\overline{p}(t) \leqslant p_0(t)$ and (with $k = 1$, $n = 0$, $t = c$) we obtain $\overline{p}(u) \leqslant c < p_0(u)$. It is not difficult to see that $\overline{p}$ is also ε-subadditive. Since $\overline{p} \leqslant p_0$, $\overline{p}(u) < p_0(u)$, hence, by the minimality of p_0, we get that $q_0 \not\leqslant \overline{p}$, that is there exists t such that $\overline{p}(t) < q_0(t)$. Thus, there are $k, n \geqslant 0$, $x_1, \ldots, x_n \in S$ such that

$$kc + \sum_{i=1}^{n} p_0(x_i) + (k+n-1)\varepsilon < q_0\left(ku + \sum_{i=1}^{n} x_i\right). \tag{1}$$

Here $k = 0$ is impossible because

$$p_0\left(\sum_{i=1}^n x_i\right) \leqslant \sum_{i=1}^n p_0(x_i) + (n-1)\varepsilon < q_0\left(\sum_{i=1}^n x_i\right)$$

is an obvious contradiction.

A similar argument shows (interchanging the roles of p_0, q_0 and ε, δ) that there exist $l \geqslant 1$, $m \geqslant 0$, $y_1, \ldots, y_m \in S$ such that

$$p_0\left(lu + \sum_{j=1}^m y_j\right) < lc + \sum_{j=1}^m q_0(y_j) - (l+m-1)\delta. \tag{2}$$

Multiplying (1) by l, (2) by k and adding up these two inequalities, we get

$$\begin{aligned} l\sum_{i=1}^n p_0(x_i) + l(k+n-1)\varepsilon + kp_0\left(lu + \sum_{j=1}^m y_j\right) \\ < k\sum_{j=1}^m q_0(y_j) - k(l+m-1)\delta + lq_0\left(ku + \sum_{i=1}^n x_i\right). \end{aligned}$$

Using the ε-subadditivity of p_0 and the δ-superadditivity of q_0, this yields

$$\begin{aligned} p_0\left(klu + l\sum_{i=1}^n x_i + k\sum_{j=1}^m y_j\right) + (k-1)(l-1)\varepsilon \\ < q_0\left(klu + l\sum_{i=1}^n x_i + k\sum_{j=1}^m y_j\right) - (k-1)(l-1)\delta \end{aligned}$$

which contradicts $q_0 \leqslant p_0$, $\varepsilon, \delta \geqslant 0$. Thus the proof is complete.

László Székelyhidi and Zsolt Páles

6. Remark. (On functional equations connected with an embedding problem). In 1997 L. Reich posed the problem which can be shortly described as follows: under what assumptions the well–known linear functional equation can be "embeded" into a "continuous time" equation prolonging in natural way its iterative process [2]. The fundamental role in solving of this problem play solutions of three functional equations in several variables

$$F(s+t, x) = F(t, F(s, x)), \tag{T}$$

$$G(s+t, x) = G(s, x)G(t, F(s, x)), \tag{G}$$

$$H(s+t,x) = G(t,F(s,x))H(s,x) + H(t,F(s,x)), \tag{H}$$

defined, in general, on the product of a commutative semigroup and a set X, with values in X, a given field $\mathbb{K}$ and a linear space Z over $\mathbb{K}$, respectively.

We present new results on solutions of equations (T) and (G) defined on the product of the halfline of positive reals and an arbitrary closed interval. Presented results are contained in the paper [1].

Let $F : (0,\infty) \times [a,b] \to [a,b]$, where $-\infty \leqslant a < b \leqslant \infty$, be an continuous iteration semigroup such that the following conditions hold

(i) the function $F(1,\cdot)$ has no fixed points in (a,b),

(ii) there is a point $c \in [a,b]$ such that the functions $F(1,\cdot)|_{[a,c]}$ and $F(1,\cdot)|_{[c,b]}$ are, respectively, strictly increasing and constant, if $F(1,x) > x$ for $x \in (a,b)$ and, respectively, constant and strictly increasing, if $F(1,x) < x$ for $x \in (a,b)$.

In the comprehensive paper [3] by M. C. Zdun one can find the collection of theorems completly describing continuous iteration semigroups, but now we present the result which is a reformulation and simplification of some Zdun's theorems.

Theorem A

Let $-\infty \leqslant a < b \leqslant \infty$ and let $F : (0,\infty) \times [a,b] \to [a,b]$ be a function satisfying conditions (i) *and* (ii)*.*

The function F is a continuous iteration semigroup if and only if there exists a continuous strictly monotonic function α mapping $[a,b]$ onto an interval $[p,q] \subset [-\infty,\infty]$ such that

$$F(t,x) = \alpha^{-1}(\min\{\alpha(x)+t, q\}) \tag{1}$$

for every $t \in (0,\infty)$ and $x \in [a,b]$.

The theorem describing general solutions of equations (G) in the case when F is a continuous iteration semigroup is following.

Theorem B

Let $-\infty \leqslant a < b \leqslant \infty$, $F : (0,\infty) \times [a,b] \to [a,b]$ be a continuous iteration semigroup satisfying conditions (i) *and* (ii) *and let α be a continuous strictly monotonic function mapping an interval $[a,b]$ onto $[p,q] \subset [-\infty,\infty]$ such that the function F is of the form* (1)*. Assume that $(Y,\cdot)$ is a commutative group.*

The function $G : (0,\infty) \times [a,b] \to Y$ is a solution of equation (G) *if and only if there exists a function $\phi : \alpha^{-1}([p,q] \cap \mathbb{R}) \to Y$ and solutions $m_1, m_2 : (0,\infty) \to Y$ of Cauchy's equation*

$$m(s+t) = m(s)m(t) \tag{M}$$

such that for every $t \in (0,\infty)$ and $x \in [a,b]$

$$G(t,x) = \begin{cases} m_1(t), & \text{if } \alpha(x)+t = p, \\ \frac{\phi(F(t,x))}{\phi(x)}, & \text{if } p < \alpha(x)+t \leqslant q \text{ and } x \neq b, \\ \frac{\phi(F(t,x))}{\phi(x)} m_2(\alpha(x)+t-q), & \text{if } \alpha(x)+t > q \text{ and } x \neq b, \\ m_2(t), & \text{if } x = b, \end{cases} \tag{2}$$

whenever α is increasing.

Note that the form of solutions of (G) in the case when the generator α is decreasing is similar.

Remark that the theorem on the form of general solution of equation (H) in the case when F is a continuous iteration semigroup, G solves (G) and both are defined on the product of $(0,\infty)$ and an arbitrary compact interval is now prepared by the present author.

[1] G. Guzik, W. Jarczyk, J. Matkowski, *Cocycles of continuous iteration semigroups*, submitted.

[2] L. Reich, *24 Remark*, in: *The Thirty-Fifth International Symposium on Functional Equations, September 7-14, 1997, Graz-Mariatrost, Austria*, Aequationes Math. **55**(1998), 311-312.

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7. Problems. (Functional equations related to bijections preserving independence)

1. Real positive random variables. Consider functions: $\gamma_{p,b} : (0,\infty) \to (0,\infty)$, $\beta_{p,q} : (0,1) \to (0,\infty)$ and $\mu_{p,a,b} : (0,\infty) \to (0,\infty)$ defined by the formulas

$$\begin{aligned} \gamma_{p,b}(x) &= C_1 x^{p-1} e^{-bx}, \\ \beta_{p,q}(x) &= C_2 x^{p-1}(1-x)^{q-1}, \\ \mu_{p,a,b}(x) &= C_3 x^{-p-1} e^{-ax-\frac{b}{x}}, \end{aligned}$$

where a, b, p, q are positive constants and C_1, C_2, C_3 are normalizing constants making the respective functions integrable to 1 over their domains. Observe that they are, respectively, probability density functions of the gamma, beta and generalized inverse Gaussian distribution (which will be denoted by the same symbols). Consider also three bijections $\psi_1 : (0,\infty)^2 \to (0,1) \times (0,\infty)$, $\psi_2 : (0,\infty)^2 \to (0,\infty)^2$, $\psi_3 : (0,1)^2 \to (0,1)^2$ defined by

$$\psi_1(x,y) = \left(\frac{x}{x+y}, x+y \right),$$

$$\psi_2(x,y) = \left(\frac{1}{x+y}, \frac{1}{x} - \frac{1}{x+y}\right),$$
$$\psi_3(x,y) = \left(\frac{1-y}{1-xy}, 1-xy\right).$$

Then it can be easily checked that these maps preserve independence in the following sence:

1° if the random vector (X,Y) has the distribution $\gamma_{p,b} \otimes \gamma_{q,b}$ ("$\otimes$" stands for the product measure), i.e. the component random variables are independent (we write it as: $(X,Y) \sim \gamma_{p,b} \otimes \gamma_{q,b}$) then $(U,V) = \psi_1(X,Y) \sim \beta_{p,q} \otimes \gamma_{p+q,b}$.

2° If $(X,Y) \sim \mu_{p,a,b} \otimes \gamma_{p,q}$ then $(U,V) = \psi_2(X,Y) \sim \mu_{p,b,a} \otimes \gamma_{p,b}$.

3° If $(X,Y) \sim \beta_{p,q} \otimes \beta_{p+q,r}$ then $(U,V) = \psi_3(X,Y) \sim \beta_{r,q} \otimes \beta_{r+p,p}$.

Consider now converse problems, i.e. assume that (X,Y) is a random vector with independent components (and a suitable range) and $(U,V) = \psi_i(X,Y)$ for a given fixed $i \in \{1,2,3\}$ has also independent components. The basic question reads: does this property of preserving independence under given ψ_i characterize respective distributions? Such a question leads to the following functional equations (if existence of densities is assumed): For $i=1$ and $(u,v) \in (0,1) \times (0,\infty)$ almost everywhere

$$f_U(u)f_V(v) = v f_X(uv) f_Y((1-u)v);$$

for $i=2$ and $(u,v) \in (0,\infty)^2$ almost everywhere

$$f_U(u)f_V(v) = \frac{1}{u^2(u+v)^2} f_X\left(\frac{1}{u+v}\right) f_Y\left(\frac{1}{u} - \frac{1}{u+v}\right);$$

for $i=3$ and $(u,v) \in (0,1)^2$ almost everywhere

$$f_U(u)f_V(v) = \frac{v}{1-uv} f_X\left(\frac{1-v}{1-uv}\right) f_Y(1-uv);$$

where f_U, f_V, f_X, f_Y are probability density functions, i.e. are non-negative and integrable to 1 over their domains. In each of these three cases the solutions are known to be of the form as in the direct results (see 1°-3° above) under additional technical assumptions that f_X and f_Y are strictly positive and their logarithms are locally integrable functions. The question lies in removing these restrictions.

2. Matrix variate random variables. Denote by $\mathcal{V}_+$ the cone of symmetric, real, positive definite $n \times n$ matrices, which is a subset of the Euclidean space $\mathcal{V}$ of symmetric, real $n \times n$ matrices with the inner product $(a, b) = \mathrm{trace}\,(ab)$. Fix the Lebesgue measure on $\mathcal{V}$ by assigning the unit mass to the unit cube. Then the matrix variate gamma, beta and generalized inverse Gaussian distributions are defined by their densities of the form:

$$\gamma_{p,b}(x) = C_1(\det x)^{p-\frac{n+1}{2}}\exp\left(-(a,x)\right), \quad x \in \mathcal{V}_+;$$

$$\beta_{p,q}(x) = C_2(\det x)^{p-\frac{n+1}{2}}(\det(e-x))^{q-\frac{n+1}{2}}, \quad x \in \mathcal{D} = \{x \in \mathcal{V}_+ : e - x \in \mathcal{V}_+\};$$

$$\mu_{p,a,b}(x) = C_3(\det x)^{-p-\frac{n+1}{2}}\exp\left(-(a,x) - (b, x^{-1})\right), \quad x \in \mathcal{V}_+$$

where $a, b \in \mathcal{V}_+$, $p, q > \frac{n-1}{2}$ and C_1, C_2, C_3 are normalizing constants (e denotes the identity matrix).

Again consider three bijections: $\psi_1 : \mathcal{V}_+^2 \to \mathcal{D} \times \mathcal{V}_+$, $\psi_2 : \mathcal{V}_+^2 \to \mathcal{V}_+^2$, $\psi_3 : \mathcal{D}^2 \to \mathcal{D}^2$ defined by

$$\begin{aligned}
\psi_1(x,y) &= \left((x+y)^{-\frac{1}{2}}x(x+y)^{-\frac{1}{2}}, x+y\right), \\
\psi_2(x,y) &= \left((x+y)^{-1}, x^{-1} - (x+y)^{-1}\right), \\
\psi_3(x,y) &= \left((e - y^{\frac{1}{2}}xy^{\frac{1}{2}})^{-\frac{1}{2}}(e-y)(e - y^{\frac{1}{2}}xy^{\frac{1}{2}})^{-\frac{1}{2}}, e - y^{\frac{1}{2}}xy^{\frac{1}{2}}\right).
\end{aligned}$$

Then, by computing respective jacobians, it can be chcked that the statements $1°$-$3°$ of the preceding section hold true (with the meaning of all the symbols as defined in this section). With the converse results the situation is much more complicated than in the univariate case. First of all assume that (X, Y) and $(U, V) = \psi_i(X, Y)$ for a given fixed $i \in \{1, 2, 3\}$ have independent components. Then for $i = 1$ we obtain the following equation for the densities:

$$f_U(u)f_V(v) = (\det v)^{\frac{n+1}{2}} f_X(v^{\frac{1}{2}}uv^{\frac{1}{2}})f_Y(v^{\frac{1}{2}}(e-u)v^{\frac{1}{2}})$$

for $(u, v) \in \mathcal{D} \times \mathcal{V}_+$ almost everywhere. This equation was solved under quite restrictive conditions that f_X and f_Y are strictly positive and twice differentiable on their domains. The proof was based on solutions of other two functional equations:

$$a(x) = g(yxy) - g(y(e-x)y), \quad (x,y) \in \mathcal{D} \times \mathcal{V}_+$$

solved under the differentiability condition assumed for g and

$$a_1(x) + a_2(y) = g(yxy) + g(y(e-x)y), \quad (x,y) \in \mathcal{D} \times \mathcal{V}_+$$

solved under the assumption that g is twice differentiable. For all the above three equations the open problem lies in reducing the apriori smoothness conditions imposed on the unknown functions. Also a related multiplicative Cauchy functional equation in $\mathcal{V}_+$ of the form

$$f(x)f(y) = f(y^{\frac{1}{2}}xy^{\frac{1}{2}}), \quad (x,y) \in \mathcal{V}_+^2$$

where $f : \mathcal{V}_+ \to (0,\infty)$ was solved under the differentiability assumption imposed on f. Here again the question lies in removing the smoothness restriction.

For $i = 2$ the equation for densities reads

$$f_U(u)f_V(v) = \frac{f_X((u+v)^{-1})f_Y(u^{-1} - (u+v)^{-1})}{(\det u \det(u+v))^{n+1}}$$

for $(u,v) \in \mathcal{V}_+^2$ almost everywhere, which was solved under the restriction that f_X and f_Y are strictly positive differentiable functions. Again the question here lies in removing these artificial assumptions.

For $i = 3$ the equation for densities is of the form

$$f_U(u)f_V(v) = \left(\frac{\det v}{\det y(u,v)}\right)^{\frac{n+1}{2}} f_Y(y(u,v))f_X(y^{-\frac{1}{2}}(u,v)(e-v)y^{-\frac{1}{2}}(u,v))$$

for $(u,v) \in \mathcal{D}^2$ almost everywhere, where $y(u,v) = e - v^{\frac{1}{2}}uv^{\frac{1}{2}}$. However in this case even a solution under additional assumption smoothness conditions is not known. Observe that if densities f_X and f_Y are strictly positive then the equation can be reduced to

$$g_1(t) + g_2(w) = g_3(w(e - t^{-2})w) + g_4(t(e - w^{-2})t)$$

for $(t,w) \in (e + \mathcal{V}_+)^2$ almost ewerywhere.

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